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PARTIAL DIFFERENTIAL EQUATIONS

On the Existence of a Continuous Branch of Nodal Solutions of Elliptic Equations with Convex-Concave Nonlinearities

V. E. Bobkov

Institute of Mathematics with Computer Center,
Ufa Scientific Center of the Russian Academy of Sciences, Ufa, Russia
e-mail: bobkovve@gmail.com

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Abstract—We study the existence of nodal solutions of a parametrized family of Dirichlet boundary value problems for elliptic equations with convex-concave nonlinearities. In the main result, we prove the existence of nodal solutions u_λ for $\lambda \in (-\infty, \lambda_0^*)$. The critical value $\lambda_0^* > 0$ is found by a spectral analysis procedure according to Pokhozhaev's fibering method. We show that the obtained solutions form a continuous branch (in the sense of level lines of the energy functional) with respect to the parameter λ. Moreover, we prove the existence of an interval $(-\infty, \tilde{\lambda})$, where $\tilde{\lambda} > 0$, on which this branch consists of solutions with exactly two nodal domains.

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INTRODUCTION

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with piecewise smooth boundary $\partial\Omega$, consider the Dirichlet problem

$$-\Delta u = \lambda k(x)|u|^{q-2}u + h(x)|u|^{\gamma-2}u, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega. \tag{1}$$

We assume that the weight functions $k(x), h(x) \in L^\infty(\Omega)$ satisfy the conditions

$$\operatorname*{ess\,inf}_{x\in\Omega} k(x) > 0, \qquad \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 0, \tag{2}$$

$\lambda \in \mathbb{R}$, and

$$1 < q < 2 < \gamma < 2^*, \qquad 2^* = \begin{cases} 2n/(n-2) & \text{if} \quad n > 2, \\ +\infty & \text{if} \quad n \leq 2. \end{cases} \tag{3}$$

Under conditions (2) and (3), the nonlinearity in the problem is said to be *convex-concave* (see [1]).

Problems of the type (1) arise in various fields of mathematical physics: in the study of anisotropic media [2], in the description of a flow on a impermeable plate [3], in the description of the superdiffusion phenomenon, etc. (see the bibliography in [2]) as well as in models of population dynamics. They play an important role in the study of systems of the activator–inhibitor type in the modeling of biological pattern formation processes [4] and can be viewed as a variant of the stationary equation for the population density flow in the Patlak–Keller–Segel chemotaxis model [5].

Starting from [1], numerous papers (e.g., see [1, 2, 6–8]) deal with the existence of positive solutions, multiple positive solutions, several solutions, and infinitely many solutions for problems of the form (1).

In addition, the interest in problems of the existence of nodal solutions of nonlinear elliptic equations [9–12] has recently been growing. This is caused by the fact that nodal solutions arise in applications [12]. Moreover, the interest in the study of nodal solutions is related to a number of well-known unsolved problems of nonlinear analysis, such as determining the number of nodal domains and geometric properties of nodal sets of solutions [13], the problem of the generalization of the Courant theorem on zeros of eigenfunctions to nonlinear problems [14, Chap. 6, item 6; 15], and the close so-called optimal partition problem [16]. Note also that well-known topological methods like the Lyusternik–Schnirelmann theorem [6] permit one to prove the existence of countably many solutions of nonlinear elliptic equations but do not provide complete information on their structure, i.e., changing of the sign, the number of nodal domains, geometric properties of nodal sets, etc.

The Nehari manifold method (see [17, 18]) is one of the most efficient and most often used methods for proving the existence of nodal solutions. At the same time, Pokhozhaev's global fibering method [19, 20], which is more efficient when studying parametrized problems, proving the existence of branches of solutions, and analyzing bifurcations, is more widely used in the Russian mathematical school.

The present paper deals with the existence of nodal solutions of problem (1). Special attention is paid to finding solutions with exact number of nodal domains (with two nodal domains) and to the construction of the corresponding continuous (in the sense of level lines of the energy functional) branch of solutions.

Note that the well-known papers on the theory of nodal solutions and solutions with exact number of nodal domains use approaches based on the fact that, in considered classes of problems, all possible solutions only correspond to nonnegative levels of the energy functional. The main difficulty in the search of nodal solutions of problem (1) is related to the fact that the solutions of problem (1) can correspond to arbitrary signs of levels of the energy functional.

Let us state the main results of the present paper. We consider *weak solutions* of problem (1), that is, functions $u \in W_0^{1,2}(\Omega)\backslash\{0\}$ that are critical points of the energy functional

$$I_\lambda(u) = \frac{1}{2}H(u) - \frac{\lambda}{q}G(u) - \frac{1}{\gamma}F(u),$$

where

$$H(u) = \int\limits_\Omega |\nabla u|^2\,dx, \qquad G(u) = \int\limits_\Omega k(x)|u|^q\,dx, \qquad F(u) = \int\limits_\Omega h(x)|u|^\gamma\,dx.$$

In particular, any weak solution u satisfies the relation

$$Q_\lambda(u) = D_u I_\lambda(u)(u) = H(u) - \lambda G(u) - F(u) = 0,$$

i.e., belongs to the *Nehari manifold*

$$\mathcal{N}_\lambda = \{v \in W_0^{1,2}(\Omega)\backslash\{0\} :\ Q_\lambda(v) = 0\}.$$

Il'yasov [6] used spectral analysis by Pokhozhaev's fibering method [19, 21], introduced the critical value λ_0^* defined by the variational problem

$$\lambda_0^* = \frac{q(\gamma-2)}{\gamma(2-q)}\left(\frac{\gamma(2-q)}{2(\gamma-q)}\right)^{(\gamma-q)/(\gamma-2)} \inf_{v\in W\backslash\{0\}}\left(\frac{H^{(\gamma-q)/(\gamma-2)}(v)}{G(v)F^{(2-q)/(\gamma-2)}(v)}\right), \tag{4}$$

and showed that, for all $\lambda \in (0, \lambda_0^*)$, the Nehari manifold consists of two disjoint components separated by the sign of the functional $\mathcal{L}_\lambda$ defined by the relation

$$\mathcal{L}_\lambda(u) = D^2_{uu} I_\lambda(u)(u,u) = H(u) - \lambda(q-1)G(u) - (\gamma-1)F(u).$$

A *nodal solution* $u \in W_0^{1,2}(\Omega)$ of problem (1) is defined as a weak solution such that

$$u_+ := \max\{u, 0\} \neq 0, \qquad u_- := \min\{0, u\} \neq 0.$$

In this case, the set $\mathcal{M} = \overline{\{x \in \Omega :\ u(x) = 0\}}$ is referred to as a *nodal set*, and the connected components (maximal connected subsets) of the set $\Omega\backslash\mathcal{M}$ are referred to as *nodal domain*s (for the terminology, see [14, p. 429; 16]).

Consider the following subset of nodal functions in $\mathcal{N}_\lambda$:

$$\mathcal{N}_\lambda^1 = \{v \in \mathcal{N}_\lambda : \ v_+ \in \mathcal{N}_\lambda, \ v_- \in \mathcal{N}_\lambda, \ L_\lambda(v_+) < 0, \ L_\lambda(v_-) < 0\}.$$

A weak solution $u_\lambda \in \mathcal{N}_\lambda^1$ of problem (1) is referred to as a *ground state* with respect to $\mathcal{N}_\lambda^1$ if

$$I_\lambda(u_\lambda) \le I_\lambda(v), \qquad v \in \mathcal{N}_\lambda^1.$$

The following assertion is the main result of the present paper.

Theorem 1. *Let* $1 < q < 2 < \gamma < 2^*$. *Then for each* $\lambda \in (-\infty, \lambda_0^*)$, *there exists a nodal solution* $u_\lambda = u_\lambda^+ + u_\lambda^-$ *of problem* (1) *such that* $u_\lambda \in \mathcal{N}_\lambda^1$. *Furthermore*, u_λ *is a ground state with respect to* $\mathcal{N}_\lambda^1$.

We say that the family of critical points u_λ of the functional I_λ forms a *continuous branch* of solutions along level lines of I_λ on the interval (a, b) if the mapping

$$I_{(\cdot)}(u_{(\cdot)}) : \ (a, b) \to \mathbb{R}$$

is a continuous function.

Theorem 2. *Let* $1 < q < 2 < \gamma < 2^*$. *Then the set of ground states* u_λ *of problem* (1) *with respect to* $\mathcal{N}_\lambda^1$ *is a continuous branch of solutions along the level lines of* I_λ *on the interval* $(-\infty, \lambda_0^*)$.

Theorem 3. *Let* $1 < q < 2 < \gamma < 2^*$. *Then there exists* $\tilde{\lambda} > 0$ *such that each ground state* u_λ *of problem* (1) *with respect to* $\mathcal{N}_\lambda^1$ *for* $\lambda \in (-\infty, \min\{\tilde{\lambda}, \lambda_0^*\})$ *has exactly two nodal domains.*

1. ANALYSIS OF THE FUNCTIONAL I_λ BY THE FIBERING METHOD

A fibration of the functional $I_\lambda(u)$, $u \in W_0^{1,2}(\Omega)$, is defined as an extended, to $\mathbb{R}^+ \times W_0^{1,2}(\Omega)$, functional $\tilde{I}_\lambda(t, u) = I_\lambda(tu)$, where $t > 0$; in addition,

$$Q_\lambda(u) = \frac{\partial}{\partial t} I_\lambda(tu)|_{t=1}, \qquad \mathcal{L}_\lambda(u) = \frac{\partial^2}{\partial t^2} I_\lambda(tu)|_{t=1}.$$

Note that the variational problem (4) for λ_0^* can be obtained from the system

$$\frac{1}{2}\, t^2 H(u) - \frac{\lambda}{q}\, t^q G(u) - \frac{1}{\gamma}\, t^\gamma F(u) = 0, \qquad tH(u) - \lambda t^{q-1} G(u) - t^{\gamma-1} F(u) = 0,$$

which corresponds to the case in which $I_\lambda(tu) = 0$ and $Q_\lambda(tu) = 0$, for an arbitrary function $u \in W_0^{1,2}(\Omega)\backslash\{0\}$. By solving this system for $\lambda = \lambda(u)$ and $t = t(u)$, we obtain

$$\lambda(u) = \frac{q(\gamma-2)}{\gamma(2-q)} \left(\frac{\gamma(2-q)}{2(\gamma-q)}\right)^{(\gamma-q)/(\gamma-2)} \frac{H^{(\gamma-q)/(\gamma-2)}(u)}{G(v) F^{(2-q)/(\gamma-2)}(u)}. \tag{5}$$

Next, following [6], from (5), we obtain the critical value (4).

The following assertion was proved in [6].

Proposition 1. *Let* $1 < q < 2 < \gamma < 2^*$, *let* $u \in W_0^{1,2}(\Omega)\backslash\{0\}$, *and let the parameter* λ_0^* *be given by the variational problem* (4). *Then the following assertions hold.*

1. *If* $\lambda \in (0, \lambda_0^*)$, *then* $I_\lambda(tu)$ *treated as a function of* t *has exactly two critical points*, *a point of minimum* $t_1(u)$ *and a point of maximum* $t_2(u)$; *moreover*, $t_1(u) < t_2(u)$.

2. *If* $\lambda \le 0$, *then* $I_\lambda(tu)$ *treated as a function of* t *has exactly one critical point*, *a point of maximum* $t_3(u)$.

Below we need the following assertions.

Lemma 1. *Let* $1 < q < 2 < \gamma < 2^*$, $\lambda < \lambda_0^*$, *and* $u \in \mathcal{N}_\lambda$. *Then*

1. $\mathcal{L}_\lambda(u) \neq 0$;

2. $I_\lambda(u) \to +\infty$ *as* $\|u\| \to +\infty$; *i.e., the functional* I_λ *is coercive on* $\mathcal{N}_\lambda$;

3. *if* $\mathcal{L}_\lambda(u) > 0$, *then* $\|u\| < \delta_1 < +\infty$, *where* δ_1 *is independent of* u; *in addition*, $\|u\| \to 0$ *and* $\mathcal{L}_\lambda(u) \to 0+$ *as* $\lambda \to 0$.

Proof. 1. Suppose that $\mathcal{L}_\lambda(u) = 0$. Then, by virtue of the relation $Q_\lambda(u) = 0$, we find that $t = 1$ is a point of inflection of the fibered functional $I_\lambda(tu)$. However, by Proposition 1, if $\lambda < \lambda_0^*$, then $I_\lambda(tu)$ treated as a function of t has no critical point of inflection type. Consequently, $\mathcal{L}_\lambda(u) \neq 0$.

2. Now let us prove the coercivity of the functional I_λ on $\mathcal{N}_\lambda$. By assumption, $Q_\lambda(u) = 0$; therefore, the functional I_λ on $\mathcal{N}_\lambda$ can be represented in the form

$$I_\lambda(u) = \frac{\gamma - 2}{2\gamma} H(u) - \lambda \frac{\gamma - q}{\gamma q} G(u). \tag{6}$$

If $\lambda > 0$, then, by the embedding theorem, we obtain the estimate

$$I_\lambda(u) > \frac{\gamma - 2}{2\gamma} H(u) - \lambda \frac{\gamma - q}{\gamma q} C_q H(u)^{q/2},$$

where $C_q = C_q(q, \gamma, \Omega) > 0$ is a constant.

If $\lambda \leq 0$, then we estimate the functional (6) as follows:

$$I_\lambda(u) \geq \frac{\gamma - 2}{2\gamma} H(u).$$

Then in both cases we have $I_\lambda(u) \to +\infty$ as $H(u) = \|u\|^2 \to +\infty$; i.e., the functional I_λ is coercive on $\mathcal{N}_\lambda$.

3. Now suppose that $u \in \mathcal{N}_\lambda$ and $\mathcal{L}_\lambda(u) > 0$. We rewrite these conditions in the form of the system

$$H(u) - \lambda G(u) - F(u) = 0, \qquad H(u) - \lambda(q-1)G(u) - (\gamma - 1)F(u) > 0.$$

By expressing $F(u)$ from the equation and by substituting it into the inequality, we obtain

$$\mathcal{L}_\lambda(u) = -(\gamma - 2)H(u) + \lambda(\gamma - q)G(u) > 0. \tag{7}$$

Next, by using the Sobolev embedding theorem, we obtain the chain of inequalities

$$H(u) < \lambda \frac{\gamma - q}{\gamma - 2} G(u) < \lambda C_q \frac{\gamma - q}{\gamma - 2} H(u)^{q/2}, \qquad C_q = C_q(q, \Omega) > 0,$$

which implies that

$$\|u\|^2 = H(u) < \left(\lambda C_q \frac{\gamma - q}{\gamma - 2}\right)^{2/(2-q)} = \delta_1^2(\lambda, q, \gamma, \Omega) = \delta_1^2. \tag{8}$$

In addition, $\|u\| < \delta_1 \to 0$ as $\lambda \to 0$. Moreover, one can readily find from inequalities (7) and (8) that $\mathcal{L}_\lambda(u) \to 0+$ as $\lambda \to 0$. The proof of the lemma is complete.

Lemma 2. *Let* $1 < q < 2 < \gamma < 2^*$ *and* $\lambda < \lambda_0^*$. *If* $u \in \mathcal{N}_\lambda$ *and* $\mathcal{L}_\lambda(u) < 0$, *then the following assertions hold.*

1. $I_\lambda(u) > 0$, *and* $t = 1$ *is a point of global maximum of* $I_\lambda(tu)$ *treated as a function of* t *for* $t > 0$.

2. $\|u\| > \delta_2 > 0$ *and* $\mathcal{L}_\lambda(u) < C < 0$, *where* δ_2 *and* C *are independent of* u *and* λ.

Proof. 1. Note that $\lambda < \lambda_0^* \le \lambda(u)$, where $\lambda(u)$ is given by relation (5); therefore,

$$I_\lambda(u) = \frac{1}{2}H(u) - \frac{\lambda}{q}G(u) - \frac{1}{\gamma}F(u) > \frac{1}{2}H(u) - \frac{\lambda(u)}{q}G(u) - \frac{1}{\gamma}F(u) = 0;$$

i.e., $I_\lambda(u) > 0$. By Proposition 1, the point $t = 1$ is the unique point of local maximum of $I_\lambda(tu)$ treated as a function of t for $t > 0$, and $I_\lambda(u) > 0$. In this case, the relations

$$I_\lambda(tu) \to 0 \quad \text{as} \quad t \to 0, \quad I_\lambda(tu) \to -\infty \quad \text{as} \quad t \to +\infty$$

hold on the boundary of the domain $(0, +\infty)$. Consequently, $t = 1$ is a point of global maximum of $I_\lambda(tu)$ treated as a function of t.

2. We rewrite the conditions $Q_\lambda(u) = 0$ and $\mathcal{L}_\lambda(u) < 0$ in the form of the system

$$H(u) - \lambda G(u) - F(u) = 0, \qquad H(u) - \lambda(q-1)G(u) - (\gamma - 1)F(u) < 0.$$

We express $\lambda G(v)$ from the equation and substitute it into the inequality. Then, by virtue of the Sobolev embedding theorem, we obtain the chain of inequalities

$$\frac{2-q}{\gamma-q}H(u) < F(u) < C_\gamma H(u)^{\gamma/2}, \qquad C_\gamma = C_\gamma(q, \gamma, \Omega) > 0,$$

which implies that

$$H(u) > \left(\frac{2-q}{(\gamma-q)C_\gamma}\right)^{2/(\gamma-2)} = \delta_2^2(q, \gamma, \Omega) = \delta_2^2 > 0.$$

Therefore, $\|u\| = H(u)^{1/2} > \delta_2 > 0$.

Now let us show that $\mathcal{L}_\lambda(u) < C < 0$, where C is independent of u and λ. By assumption, $\mathcal{L}_\lambda(u) < 0$, and it follows from the assertions proved above that $I_\lambda(u) > 0$. We rewrite these inequalities in the form of the system

$$I_\lambda(u) = -\frac{2-q}{2}H(u) + \frac{\gamma-q}{\gamma}F(u) > 0, \qquad \mathcal{L}_\lambda(u) = (2-q)H(u) - (\gamma-q)F(u) < 0,$$

which implies that

$$\mathcal{L}_\lambda(u) < -\frac{(2-q)(\gamma-2)}{2}H(u) < C(q, \gamma, \Omega) = C < 0,$$

because $\|u\| > \delta_2 > 0$ as was shown above. The proof of the lemma is complete.

Note that if $u \in W_0^{1,2}(\Omega)$, then $u_\pm \in W_0^{1,2}(\Omega)$ (see Corollary A.5 in [22, p. 54]), and for the representation $u = u_+ + u_-$, we have the relations

$$I_\lambda(u) = I_\lambda(u_+) + I_\lambda(u_-), \qquad Q_\lambda(u) = Q_\lambda(u_+) + Q_\lambda(u_-), \qquad \mathcal{L}_\lambda(u) = \mathcal{L}_\lambda(u_+) + \mathcal{L}_\lambda(u_-).$$

Remark 1. The assertions of Lemmas 1 and 2 remain valid for u_+ and u_- if $u \in \mathcal{N}_\lambda^1$.

2. EXISTENCE OF NODAL SOLUTIONS

First, we show that if $\lambda < \lambda_0^*$, then the set $\mathcal{N}_\lambda^1$ is nonempty. Take an arbitrary subdomain $\Omega_1 \subset \Omega$ and a function $u_1 \in W_0^{1,2}(\Omega)\backslash\{0\}$ such that $\operatorname{supp} u_1 \subseteq \overline{\Omega}_1$. Then, by Proposition 1, there exists a $t_2(u_1) > 0$ such that $Q_\lambda(t_2(u_1)u_1) = 0$ and $\mathcal{L}_\lambda(t_2(u_1)u_1) < 0$. Now take some subdomain $\Omega_2 \subset \Omega$ such that $\Omega_1 \cap \Omega_2 = \varnothing$ and a function $u_2 \in W_0^{1,2}(\Omega)\backslash\{0\}$ such that $\operatorname{supp} u_2 \subseteq \overline{\Omega}_2$. Then there exists a $t_2(u_2) > 0$ such that $Q_\lambda(t_2(u_2)u_2) = 0$ and $\mathcal{L}_\lambda(t_2(u_2)u_2) < 0$. Set $v_+ = t_2(u_1)u_1$, $v_- = -t_2(u_2)u_2$, and $v = v_+ + v_-$. Then $Q_\lambda(v) = Q_\lambda(v_+) + Q_\lambda(v_-) = 0$.

We have thereby found a function $v \in W_0^{1,2}(\Omega)\backslash\{0\}$ that belongs to the set $v \in \mathcal{N}_\lambda^1$; i.e., $\mathcal{N}_\lambda^1 \ne \varnothing$.

Next, consider the constrained minimization problem

$$I_\lambda(u) \to \min, \qquad u \in \mathcal{N}_\lambda^1. \tag{9}$$

Let $c_\lambda = \inf\{I_\lambda(v) : \ v \in \mathcal{N}_\lambda^1\}$, and let $u_n \in \mathcal{N}_\lambda^1$ be a minimizing sequence; i.e., $I_\lambda(u_n) \to c_\lambda$. Note that $c_\lambda \geq 0$ by Lemma 2. Then it follows from the coercivity of I_λ on $\mathcal{N}_\lambda$ (see Lemma 1) that the sequence u_n is bounded in $W_0^{1,2}(\Omega)$. Since the space $W_0^{1,2}(\Omega)$ is reflexive, it follows from the Eberlein–Smulian theorem [23, p. 466 of the Russian translation] that there exist functions $u, v, w \in W_0^{1,2}(\Omega)$ such that

$$u_n \rightharpoonup u, \quad (u_n)_+ \rightharpoonup v, \quad (u_n)_- \rightharpoonup w \quad \text{weakly in} \quad W_0^{1,2}(\Omega).$$

Moreover, retaining the preceding numbering in n, from the Sobolev embedding theorem, we obtain

$$u_n \to u, \quad (u_n)_+ \to v, \quad (u_n)_- \to w \quad \text{in} \quad L^\gamma(\Omega) \quad \text{and} \quad L^q(\Omega), \tag{10}$$

because $q < \gamma < 2^*$.

Let us introduce the mapping $h : \ L^r \to L^r$ by the rule $h(u) = u_+$. It follows from Lemma A.1 (see below) that if $r = q$ and $r = \gamma$, then h is continuous; therefore, by relations (10), $u_+ = v \geq 0$ and $u_- = w \leq 0$. Let us show that u changes sign; i.e., $u_+ > 0$ and $u_- < 0$. Since $u_n \in \mathcal{N}_\lambda^1$, it follows from assertion 2 of Lemma 2 that

$$\lambda \int_\Omega (u)_+^q \, dx + \int_\Omega (u)_+^\gamma \, dx = \lim_{n\to\infty} \left(\lambda \int_\Omega (u_n)_+^q \, dx + \int_\Omega (u_n)_+^\gamma \, dx \right) = \lim_{n\to\infty} \|(u_n)_+\|^2 > \delta_2^2 > 0.$$

Consequently, $u_+ > 0$. In a similar way, one can show that $u_- < 0$.

Now let us show that $(u_n)_\pm \to u_\pm$ in $W_0^{1,2}(\Omega)$. It follows from the weak convergence $(u_n)_\pm \rightharpoonup u_\pm$ in $W_0^{1,2}(\Omega)$ that $\|u_\pm\|^2 \leq \liminf_{n\to\infty} \|(u_n)_\pm\|^2$. Let us show that the equality takes place. Suppose the contrary: $\|u_\pm\|^2 < \liminf_{n\to\infty} \|(u_n)_\pm\|^2$. Then

$$\|u_\pm\|^2 - \lambda G(u_\pm) - F(u_\pm) < \liminf_{n\to\infty} (\|(u_n)_\pm\|^2 - \lambda G((u_n)_\pm) - F((u_n)_\pm)) = 0.$$

Let $\lambda \in (-\infty, \lambda_0^*)$. By Proposition 1, there exists an $\alpha = t_2(u_+) > 0$ and a $\beta = t_2(u_-) > 0$ such that

$$Q_\lambda(\alpha u_+) = 0, \qquad Q_\lambda(\beta u_-) = 0, \qquad \mathcal{L}_\lambda(\alpha u_+) < 0, \qquad \mathcal{L}_\lambda(\beta u_-) < 0.$$

It follows that we have $Q_\lambda(\alpha u_+ + \beta u_-) = 0$. Then, by assumption,

$$I_\lambda(\alpha u_+ + \beta u_-) < \liminf_{n\to\infty} (I_\lambda(\alpha (u_n)_+ + \beta (u_n)_-)) = \liminf_{n\to\infty} (I_\lambda(\alpha (u_n)_+) + I_\lambda(\beta (u_n)_-)). \tag{11}$$

In turn, since $u_n \in \mathcal{N}_\lambda^1$, it follows from Remark 1 that $t = 1$ is a point of global maximum of the functions $I_\lambda(tu_+)$ and $I_\lambda(tu_-)$ with respect to t. Consequently,

$$\liminf_{n\to\infty} (I_\lambda(\alpha (u_n)_+) + I_\lambda(\beta (u_n)_-)) \leq \liminf_{n\to\infty} (I_\lambda((u_n)_+) + I_\lambda((u_n)_-)). \tag{12}$$

At the same time, we have

$$\liminf_{n\to\infty} (I_\lambda((u_n)_+) + I_\lambda((u_n)_-)) = \liminf_{n\to\infty} I_\lambda(u_n) = \inf_{\mathcal{N}_\lambda^1} I_\lambda = c_\lambda. \tag{13}$$

Therefore, it follows from relations (11)–(13) that $I_\lambda(\alpha u_+ + \beta u_-) < c_\lambda$, which contradicts the assumptions. Consequently, $(u_n)_+ \to u_+$, $(u_n)_- \to u_-$ in W, and $\alpha = \beta = 1$.

Therefore, $u \in \mathcal{N}_\lambda^1$ and $I_\lambda(u) = \inf\{I_\lambda(v) : \ v \in \mathcal{N}_\lambda^1\}$.

Now let us show that the u thus found is a critical point of the functional I_λ and hence a solution of problem (1).

Lemma 3. *Let $1 < q < 2 < \gamma < 2^*$ and $\lambda < \lambda_0^*$. If $u \in \mathcal{N}_\lambda^1$ is a solution of the minimization problem* (9), *then it is a critical point of I_λ on $W_0^{1,2}(\Omega)$; i.e.,*

$$D_u I_\lambda(u)(\phi) = 0, \qquad \phi \in W_0^{1,2}(\Omega)\backslash\{0\}.$$

Proof. Let $u \in \mathcal{N}_\lambda^1$ be a solution of the minimization problem (9); i.e., $I_\lambda(u) = c_\lambda = \inf\{I_\lambda(v) : v \in \mathcal{N}_\lambda^1\}$. Suppose the contrary: $D_u I_\lambda(u) \neq 0$.

Since $\lambda < \lambda_0^*$, it follows from Lemma 2 that $t = 1$ is a point of global maximum of $I_\lambda(tu)$ treated as a function of t. Moreover, by Remark 1, $t = 1$ is also a point of global maximum of $I_\lambda(tu_+)$ and $I_\lambda(tu_-)$ treated as functions of t. Consequently,

$$I_\lambda(su_+ + tu_-) = I_\lambda(su_+) + I_\lambda(tu_-) < I_\lambda(u_+) + I_\lambda(u_-) = I_\lambda(u_+ + u_-) = I_\lambda(u) \tag{14}$$

for all $(s,t) \in \mathbb{R}_+^2\backslash\{1,1\}$.

Since, by assumption, $D_u I_\lambda(u) \neq 0$, it follows from the continuity of the functional $D_u I_\lambda$ that there exist $\alpha, \delta > 0$ such that $\|D_u I_\lambda(v)\| \geq \alpha$ for $v \in U_{3\delta}(u) = \{w \in W_0^{1,2}(\Omega) : \|u - w\| < 3\delta\}$.

Let us introduce the function

$$g : \ A = \left(\frac{1 - t_1(u_+)}{2}, \frac{1 + t_1(u_+)}{2}\right)\left(\frac{1 - t_1(u_-)}{2}, \frac{1 + t_1(u_-)}{2}\right) \to W_0^{1,2}(\Omega),$$
$$g(s,t) = su_+ + tu_-.$$

It follows from Proposition 1 and the condition $\lambda < \lambda_0^*$ that $t_1(u_+), t_1(u_-) < 1$, consequently, $A \neq \varnothing$. Moreover, from inequality (14), we obtain the estimate

$$\beta_0 := \max_{(s,t)\in\partial A} I_\lambda(g(s,t)) < c_\lambda.$$

Set $\varepsilon := \min\{(c_\lambda - \beta_0)/2, \alpha\delta/8\}$ and $S = U_\delta(u)$. Then it follows from the deformation lemma (see Theorem A.1 in Appendix A) that there exists a homotopy $\eta \in \mathcal{C}([0,1] \times W_0^{1,2}(\Omega), W_0^{1,2}(\Omega))$ such that the following relations hold for $\eta_1 := \eta(1,\cdot)$.

1. $\eta_1(v) = v$ if $I_\lambda(v) \leq c_\lambda - 2\varepsilon$.
2. $\eta_1(\{v \in S : \ I_\lambda(v) \leq c_\lambda + \varepsilon\}) \subset \{v \in W_0^{1,2}(\Omega) : \ I_\lambda(v) \leq c_\lambda - \varepsilon\}$.
3. $I_\lambda(\eta_1(v)) \leq I_\lambda(v)$ for all $v \in W_0^{1,2}(\Omega)$.

It follows from the inclusion 2 that

$$\max_{\{(s,t)\in A:\, g(s,t)\in S\}} I_\lambda(\eta_1(g(s,t))) < c_\lambda. \tag{15}$$

On the other hand, from the inequalities 3 and (14), we obtain

$$\max_{\{(s,t)\in A:\, g(s,t)\notin S\}} I_\lambda(\eta_1(g(s,t))) \leq \max_{\{(s,t)\in A:\, g(s,t)\notin S\}} I_\lambda(g(s,t)) < c_\lambda. \tag{16}$$

For convenience, set

$$f(s,t) := \eta_1(g(s,t)).$$

Then it follows from relation 1 that $f(s,t) = g(s,t)$ for $(s,t) \in \partial A$ by virtue of the choice of ε. Consider the mapping

$$\psi : \ A \to \mathbb{R}^2, \qquad \psi(s,t) := (Q_\lambda(f(s,t)_+), Q_\lambda(f(s,t)_-)).$$

Note that $\psi(s,t) = (0,0)$ if and only if $f(s,t)_+, f(s,t)_- \in \mathcal{N}_\lambda$.

Since $f = g$ for $(s,t) \in \partial A$, we have

$$\psi(s,t) = (Q_\lambda(su_+), Q_\lambda(tu_-)), \qquad (s,t) \in \partial A;$$

in addition,

$$Q_\lambda\left(\frac{1-t_1(u_+)}{2}u_+\right) > 0, \qquad Q_\lambda\left(\frac{1+t_1(u_+)}{2}u_+\right) < 0, \tag{17}$$

$$Q_\lambda\left(\frac{1-t_1(u_-)}{2}u_-\right) > 0, \qquad Q_\lambda\left(\frac{1+t_1(u_-)}{2}u_-\right) < 0. \tag{18}$$

Then, by Theorem A.2 in Appendix A, there exists a point $(s_0, t_0) \in A$ such that $\psi(s_0, t_0) = (0, 0)$; consequently, $f(s_0, t_0)_+, f(s_0, t_0)_- \in \mathcal{N}_\lambda$. Moreover, it follows from inequalities (17) and (18) and Proposition 1 that $\mathcal{L}_\lambda(f(s_0, t_0)_+) < 0$ and $\mathcal{L}_\lambda(f(s_0, t_0)_-) < 0$, because there exists a unique point of the maximum of the functions $I_\lambda(zf(s_0, t_0)_+)$ and $I_\lambda(zf(s_0, t_0)_-)$ with respect to z for $z > 0$.

Therefore, $f(s_0, t_0) \in \mathcal{N}_\lambda^1$; i.e., $f(s_0, t_0)$ is an admissible function in the minimization problem (9). In addition, it follows from inequalities (15) and (16) that

$$I_\lambda(f(s_0, t_0)) < c_\lambda = \inf\{I_\lambda(v) : \ v \in \mathcal{N}_\lambda^1\};$$

i.e., we have arrived at a contradiction. Consequently, $D_u I_\lambda(u) = 0$; i.e., u is a critical point of I_λ on $W_0^{1,2}(\Omega)$. The proof of the lemma is complete.

Therefore, the desired $u \in \mathcal{N}_\lambda^1$ is a ground state of problem (1) with respect to $\mathcal{N}_\lambda^1$.

3. EXISTENCE OF A BRANCH OF SOLUTIONS

Let $u_\lambda \in \mathcal{N}_\lambda^1$ be a solution of the minimization problem (9). Set

$$c_\lambda := I_\lambda(u_\lambda), \qquad c_\lambda^+ := I_\lambda((u_\lambda)_+), \qquad c_\lambda^- := I_\lambda((u_\lambda)_-).$$

One can readily show that, for each λ and each sequence $\{\lambda_i\} \to \lambda$, the corresponding sequence $\{u_{\lambda_i}\}$ is a Paley–Smale sequence; i.e.,

$$I_\lambda(u_{\lambda_i}) \to c, \qquad D_u I_\lambda(u_{\lambda_i}) \to 0,$$

where $c > 0$. Then, by Proposition 8.1 in [6], there exists a $u_\lambda^0 \in \mathcal{N}_\lambda^1$ such that $u_\lambda \to u_\lambda^0$ strongly in $W_0^{1,2}(\Omega)$. In addition, obviously, u_λ^0 is a critical point of the functional I_λ.

Now let us show that u_λ^0 is also a solution of the minimization problem (9). Suppose the contrary: $c_\lambda < I_\lambda(u_\lambda^0)$; in this case, we have $\delta^+ + \delta^- > 0$, where

$$\delta^+ := I_\lambda((u_\lambda^0)_+) - c_\lambda^+, \qquad \delta^- := I_\lambda((u_\lambda^0)_-) - c_\lambda^-.$$

First, we show that the point of maximum $t_{2,+}^{\lambda+\Delta\lambda} = t_2^{\lambda+\Delta\lambda}((u_\lambda)_+)$ of the fibered functional $I_{\lambda+\Delta\lambda}(u_+)$ tends to unity as $\Delta\lambda \to 0$. To this end, consider the function $r(t, \Delta\lambda) \in C^1(\mathbb{R}^+ \times \mathbb{R})$ given by the relation

$$r(t, \Delta\lambda) := Q_{\lambda+\Delta\lambda}(t(u_\lambda)_+), \qquad (t, \Delta\lambda) \in \mathbb{R}^+ \times \mathbb{R}.$$

Note that $r(1, 0) = 0$, because $u_\lambda \in \mathcal{N}_\lambda^1$. In addition, $\mathcal{L}_\lambda((u_\lambda)_+) < C_1 < 0$ by virtue of assertion 2 of Lemma 2; therefore, there exists a neighborhood of the point $(1, 0)$ in which

$$\frac{\partial}{\partial t} r(t, \Delta\lambda) = \frac{\partial}{\partial t} Q_{\lambda+\Delta\lambda}(t(u_\lambda)_+) = \mathcal{L}_{\lambda+\Delta\lambda}(t(u_\lambda)_+) + Q_{\lambda+\Delta\lambda}(t(u_\lambda)_+) < 0;$$

i.e., the function $r(t, \Delta\lambda)$ is strictly monotone in t in the above-mentioned neighborhood. Then, by the implicit function theorem, there exist open domains U and V with $(1, 0) \in U \times V$ and a continuous function $s : \ U \to V$ satisfying the condition $s(0) = 1$; and the relation $r(t, \Delta\lambda) = 0$ is equivalent to the relation $t = s(\Delta\lambda)$, $(t, \Delta\lambda) \in U \times V$. Since

$$r(t_{2,+}^{\lambda+\Delta\lambda}, \Delta\lambda) = Q_{\lambda+\Delta\lambda}(t_{2,+}^{\lambda+\Delta\lambda}(u_\lambda)_+) = 0,$$

we have $t_{2,+}^{\lambda+\Delta\lambda} = s(\Delta\lambda) \to 1$ as $\Delta\lambda \to 0$.

By using this fact and the strong convergence $u_{\lambda_i} \to u_\lambda^0$, one can readily show that, for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that the inequalities

$$|I_{\lambda+\Delta\lambda}(t_{2,+}^{\lambda+\Delta\lambda}(u_\lambda)_+) - I_\lambda((u_\lambda)_+)| < \varepsilon, \qquad |I_\lambda((u_\lambda^0)_+) - I_{\lambda+\Delta\lambda}((u_{\lambda+\Delta\lambda})_+)| < \varepsilon$$

hold for all $|\Delta\lambda| < \delta$. From these estimates, we obtain

$$I_{\lambda+\Delta\lambda}(t_{2,+}^{\lambda+\Delta\lambda}(u_\lambda)_+) < I_{\lambda+\Delta\lambda}((u_{\lambda+\Delta\lambda})_+) + 2\varepsilon - \delta^+. \tag{19}$$

By proceeding in a similar way for $(u_\lambda)_-$, we obtain

$$I_{\lambda+\Delta\lambda}(t_{2,-}^{\lambda+\Delta\lambda}(u_\lambda)_-) < I_{\lambda+\Delta\lambda}((u_{\lambda+\Delta\lambda})_-) + 2\varepsilon - \delta^-. \tag{20}$$

From inequalities (19) and (20) and the assumption $\delta^+ + \delta^- > 0$, we find that the function $v_\lambda = t_{2,+}^{\lambda+\Delta\lambda}(u_\lambda)_+ + t_{2,-}^{\lambda+\Delta\lambda}(u_\lambda)_-$ satisfies the inequality

$$I_{\lambda+\Delta\lambda}(v_\lambda) < I_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda})$$

for sufficiently small $\varepsilon > 0$. By construction, $v_\lambda \in \mathcal{N}_\lambda^1$; consequently, we have obtained a contradiction, because $u_{\lambda+\Delta\lambda}$ minimizes $I_{\lambda+\Delta\lambda}$ on $\mathcal{N}_\lambda^1$.

Therefore, the ground states u_λ of problem (1) with respect to $\mathcal{N}_\lambda^1$ form a continuous branch along the level lines of I_λ on the interval $(-\infty, \lambda_0^*)$.

4. EXISTENCE OF EXACTLY TWO NODAL DOMAINS

Along with $\mathcal{N}_\lambda^1$, consider another set of nodal functions,

$$\mathcal{N}_\lambda^2 = \{v \in \mathcal{N}_\lambda : \; v_+ \in \mathcal{N}_\lambda, \; v_- \in \mathcal{N}_\lambda, \; L_\lambda(v_+) > 0, \; L_\lambda(v_-) < 0, \; L_\lambda(v) > 0\}.$$

Lemma 4. *Let $1 < q < 2 < \gamma < 2^*$. Then there exists a $\tilde{\lambda} > 0$ such that $\mathcal{N}_\lambda^2 = \varnothing$ for all $\lambda < \tilde{\lambda}$.*

Proof. Let $\lambda \le 0$. Since, by Proposition 1, $I_\lambda(tu)$ treated as a function of t has only one critical point, which is a point of maximum, we have $\mathcal{L}_\lambda(u) < 0$ for any function $u \in \mathcal{N}_\lambda$. Consequently, $\mathcal{N}_\lambda^2 = \varnothing$.

Let $\lambda > 0$. Suppose the contrary: there exists a sequence $\lambda_n \to 0$ as $n \to \infty$ and the corresponding sequence $(w_{\lambda_n}) \in \mathcal{N}_{\lambda_n}^2$. Then $\mathcal{L}_{\lambda_n}(w_{\lambda_n}) \to 0+$ and $\mathcal{L}_{\lambda_n}((w_{\lambda_n})_+) \to 0+$ as $n \to \infty$ by assertion 3 of Lemma 1. At the same time, it follows from assertion 2 of Lemma 2 that $\mathcal{L}_{\lambda_n}((w_{\lambda_n})_-) < C_1 < 0$. Therefore, there exists an $N > 0$ such that the inequality

$$\mathcal{L}_{\lambda_n}(w_{\lambda_n}) = \mathcal{L}_{\lambda_n}((w_{\lambda_n})_+) + \mathcal{L}_{\lambda_n}((w_{\lambda_n})_-) \le 0$$

holds for all $n > N$; in addition, $(w_{\lambda_n})_+ \ne 0$ and $(w_{\lambda_n})_- \ne 0$. Then $w_{\lambda_n} \notin \mathcal{N}_{\lambda_n}^2$, which contradicts the assumption that $w_{\lambda_n} \in \mathcal{N}_{\lambda_n}^2$. The proof of the lemma is complete.

Let us present some definitions in [16]. A *strong partition* of the set Ω is defined as a family $D := \bigcup_{i=1}^k D_i$ of pairwise disjoint sets D_i such that $\bigcup_{i=1}^k D_i \subset \Omega$ and $\mathrm{Int}(\overline{\bigcup_{i=1}^k D_i}) \backslash \partial\Omega = \Omega$. If the D_i are open connected sets, then the partition is said to be *open* and *connected*. Sets D_i and D_j are *neighboring* if the set $\mathrm{Int}(\overline{D_i \cup D_j}) \backslash \partial\Omega$ is connected.

To each partition D we assign the graph $G(D)$ whose vertices correspond to the sets D_i and whose edges are the pairs (D_i, D_j) of neighbors. The graph $G(D)$ is an undirected graph without multiple edges and loops. The graph is *bipartite* if it can be colored into two colors.

Let $u \in W_0^{1,2}(\Omega)$ be a weak solution of problem (1). It is well known (see [24, Th. 1.16, p. 11]) that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ in this case. The family $D(u) := \bigcup_{i=1}^k D_i(u)$ of nodal domains of a solution u is referred to as a *nodal partition of the solution* u.

The following assertion was proved in [16].

Proposition 2. *A nodal partition $D(u)$ of a weak solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of problem* (1) *is strong*, *open*, *and connected*, *and the corresponding graph $G(D(u))$ is bipartite.*

The following assertion is the main result of this section.

Lemma 5. *Let $1 < q < 2 < \gamma < 2^*$ and $\lambda < \min\{\tilde{\lambda}, \lambda_0^*\}$, where $\tilde{\lambda}$ is defined in Lemma* 4. *If $u \in \mathcal{N}_\lambda^1$ is a weak nodal solution of problem* (1) *that is a solution of the minimization problem* (9), *then it has exactly two nodal domains.*

Proof. As was mentioned above, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Suppose the contrary: the nodal partition $D(u)$ contains more than two connected components. Without loss of generality, we assume that the function u has three nodal domains D_1, D_2, and D_3; moreover, $u > 0$ in the domains D_1 and D_2. Set

$$u_i(x) = \begin{cases} u(x) & \text{for} \quad x \in D_i, \\ 0 & \text{for} \quad x \in \Omega \backslash D_i, \ i = 1, 2, 3. \end{cases}$$

Without loss of generality, we also assume that $\mathcal{L}_\lambda(u_2) < 0$. Let us introduce the function

$$v = -2u_1 + u = -u_1 + u_2 + u_3.$$

In this case, we have $v_+ = u_2$, $v_- = -u_1 + u_3$, and

$$\begin{aligned} Q_\lambda(v_+) &= Q_\lambda(u_2) = 0, \qquad & Q_\lambda(v_-) &= Q_\lambda(u_1) + Q_\lambda(u_3) = 0, \\ \mathcal{L}_\lambda(v_+) &= \mathcal{L}_\lambda(u_2) < 0, \qquad & \mathcal{L}_\lambda(v_-) &= \mathcal{L}_\lambda(u_1) + \mathcal{L}_\lambda(u_3). \end{aligned}$$

Suppose that $\mathcal{L}_\lambda(u_1) > 0$ and

$$\mathcal{L}_\lambda(u_+) = \mathcal{L}_\lambda(u_1) + \mathcal{L}_\lambda(u_2) < 0, \qquad \mathcal{L}_\lambda(v_-) = \mathcal{L}_\lambda(u_1) + \mathcal{L}_\lambda(u_3) > 0.$$

Then $v \in \mathcal{N}_\lambda^2$, which is impossible, because $\mathcal{N}_\lambda^2 = \varnothing$ for $\lambda < \min\{\tilde{\lambda}, \lambda_0^*\}$ by Lemma 4.

Consequently, either $\mathcal{L}_\lambda(u_1) > 0$ and

$$\mathcal{L}_\lambda(u_+) = \mathcal{L}_\lambda(u_1) + \mathcal{L}_\lambda(u_2) < 0, \qquad \mathcal{L}_\lambda(v_-) = \mathcal{L}_\lambda(u_1) + \mathcal{L}_\lambda(u_3) < 0,$$

or $\mathcal{L}_\lambda(u_1) < 0$.

We have $I_\lambda(v) = I_\lambda(u) = \inf\{I_\lambda(w) : \ w \in \mathcal{N}_\lambda^1\}$ in both cases. Then from Lemma 3, we find that $v \in \mathcal{N}_\lambda^1$ is a solution of problem (1). It follows from Proposition 2 that the graph $G(D(v))$ of the nodal partition $D(v)$ of the solution v is bipartite, but this is impossible, because the graph $G(D(u))$ of the nodal partition $D(u)$ of the solution u is bipartite as well. Consequently, u contains exactly two connected components. The proof of the lemma is complete.

APPENDIX A

Define a mapping $h : \ L^r \to L^r$, $r \geq 1$, by the relation $h(u) = \max\{u, 0\}$.

Lemma A.1. *The mapping h is continuous.*

Proof. Let $u \in L^r(\Omega)$. Then, obviously, $u_+ \in L^r(\Omega)$ and $u_- \in L^r(\Omega)$. Note that, almost everywhere in Ω, the mapping h can be represented in the form $h(u) = j(u)u$, where $j(u) = 1$ if $u \geq 0$ and $j(u) = 0$ if $u < 0$. Let $u_n \to u$ in $L^r(\Omega)$. Therefore, there exists a subsequence u_{n_k} such that $u_{n_k} \to u$ almost everywhere in Ω. Without loss of generality, for brevity, we retain the previous numbering with respect to n without passage to a subsequence. Then

$$\|h(u_n) - h(u)\|_r^r = \int\limits_\Omega |h(u_n) - h(u)|^r \, dx = \int\limits_\Omega |j(u_n)(u_n - u) + (j(u_n) - j(u))u|^r \, dx.$$

Note that since $\varphi(s) = s^r$ is a convex function for $r \geq 1$ and $s \geq 0$, it follows from the Jensen inequality $(s_1 + s_2)^r \leq 2^{r-1}(s_1^r + s_2^r)$ that

$$\|h(u_n) - h(u)\|_r^r \leq 2^{r-1}\left(\int\limits_\Omega |(u_n - u)|^r \, dx + \int\limits_\Omega |(j(u_n) - j(u))u|^r \, dx \right).$$

The first integral converges to zero, because $u_n \to u$ in $L^r(\Omega)$. On the other hand, the relation $j(u_n) \to j(u) = 0$ or $j(u_n) \to j(u) = 1$ holds for almost all $x \in \Omega$. Hence it follows that

$$\int\limits_\Omega |(j(u_n) - j(u))u|^r \, dx \leq \sup_{x\in\Omega}(j(u_n) - j(u)) \int\limits_\Omega |u|^r \, dx \to 0$$

by virtue of the inclusion $u \in L^r(\Omega)$. Therefore, $\|h(u_n) - h(u)\|_r \to 0$. Consequently, $h \in C(L^r; L^r)$. The proof of the lemma is complete.

The following assertion can be proved in a similar way.

Corollary A.1. *The inclusion* $h \in C(W_0^{1,2}(\Omega); W_0^{1,2}(\Omega))$ *holds.*

The following assertion is a version of the deformation lemma.

Theorem A.1. *Let* X *be a Banach space, and let* $I \in \mathcal{C}^1(X, \mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon > 0$, *and* $\delta > 0$ *satisfy the relation*

$$\|D_u I(u)\|_{X^*} \geq \frac{8\varepsilon}{\delta}, \qquad u \in I^{-1}([c - 2\varepsilon, c + \varepsilon]) \cap S_{2\delta},$$

where $S_{2\delta} = \{v \in X : \ \operatorname{dist}(v, S) \leq 2\delta\}$. *Then there exists a homotopy* $\eta \in \mathcal{C}([0,1] \times X, X)$ *such that*

if either $t = 0$ *or* $u \notin I^{-1}([c - 2\varepsilon, c + \varepsilon]) \cap S_{2\delta}$, *then* $\eta(t, u) = u$;

$\eta(1, \{v \in S : \ I(v) \leq c + \varepsilon\}) \subset \{v \in W : \ I \leq c - \varepsilon\}$;

the function $\eta(1, \cdot)$ *defines a homeomorphism* $X \to X$ *for all* $t \in [0, 1]$;

$\|\eta(t, u) - u\|_X \leq \delta$ *for arbitrary* $u \in X$ *and* $t \in [0, 1]$;

$I(\eta(\cdot, u))$ *is nonincreasing for any* $u \in X$;

$I(\eta(t, u)) < c$ *for all* $u \in I^{-1}((-\infty, c]) \cap S_\delta$, $t \in [0, 1]$.

Proof. The proof of this assertion can be found in [25, Lemma 2.3, p. 38].

The following assertion is a two-dimensional version of the Miranda theorem (e.g., see [26]).

Theorem A.2. *Let* $A = \{x \in \mathbb{R}^2 : \ a_1 \leq x_1 \leq b_1, \ a_2 \leq x_2 \leq b_2\}$, *let* $\psi = (\psi_1, \psi_2) : \ A \to \mathbb{R}^2$ *be a continuous mapping, and let*

$$\begin{aligned} \psi_1(a_1, x_2) \geq 0 \geq \psi_1(b_1, x_2), \qquad x_2 \in (a_2, b_2), \\ \psi_2(x_1, a_2) \geq 0 \geq \psi_2(x_1, b_2), \qquad x_1 \in (a_1, b_1). \end{aligned}$$

Then there exists a point $(x_1^0, x_2^0) \in A$ *such that* $\psi(x_1^0, x_2^0) = (0, 0)$.

Proof. The proof of this assertion can be found in [26].

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